Lecture 39

Back in chapter 14, we used the gradient vector to find the tangent plane of a surface at a point. If we only have a parametrized surface, this won't work because there is no gradient. However, it is easy to find tangent vectors to a parametrized surface. Let's say we have the surface S parametrized by

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle, \quad (u,v) \in D,$$

and we want the tangent plane at a point $P_0 = \vec{r}(u_0, v_0)$ on S. Remember from before that we get tangent vectors by taking curves in the surface, then taking their derivatives. A parametrization yields two natural curves in S through P_0:

$$C_1 : \vec{r}(u_0, v) \quad \left(\begin{array}{l}\text{holding } u \text{ constant at } u_0 \text{, and} \\ \text{letting } v \text{ vary.}\end{array}\right)$$

and

$$C_2 : \vec{r}(u, v_0) \quad \left(\begin{array}{l}\text{holding } v \text{ constant at } v_0 \text{, and} \\ \text{letting } u \text{ vary.}\end{array}\right)$$

These are called <u>grid curves</u> by the book.

Taking derivatives of C_1 & C_2 and evaluating at v_0 and u_0, respectively we have the two tangent vectors

$$\left.\frac{\partial}{\partial v}\right|_{v=v_0} \vec{r}(u_0, v) = \vec{r}_v(u_0, v_0)$$

&

$$\left.\frac{\partial}{\partial u}\right|_{u=u_0} \vec{r}(u, v_0) = \vec{r}_u(u_0, v_0).$$

These are two tangent vectors to S at P_0, and so we hope to make a normal vector from them. If $\vec{r}_u(u_0,v_0) \times \vec{r}_v(u_0,v_0) \neq \vec{0}$, then the surface S is called <u>smooth</u> at P_0, and the tangent plane at P_0 indeed has $\vec{r}_u(u_0,v_0) \times \vec{r}_v(u_0,v_0)$ as its normal vector. Note that we could always just parametrise the tangent plane by:

$$T_{P_0} S(s,t) = \overrightarrow{OP_0} + s\,\vec{r}_u(u_0,v_0) + t\,\vec{r}_v(u_0,v_0).$$

<u>Ex</u>: Find the tangent plane to the surface parametrized by $\vec{r}(u,v) = \langle u^2 - v^2, u+v, u^2+3v \rangle$ at the point $(3,1,1)$.

Sol:

We begin by finding the point (u_0, v_0) s.t.

$$\vec{r}(u_0, v_0) = \langle 3, 1, 1 \rangle : \quad \begin{cases} u^2 - v^2 = 3 & ① \\ u + v = 1 & ② \\ u^2 + 3v = 1 & ③ \end{cases}$$

①: $u^2 - v^2 = (u-v)\underbrace{(u+v)}_{②} = (u-v)(1) = u - v = 3 \Rightarrow u = 3+v$

②: $u+v = (3+v)+v = 3+2v = 1 \Rightarrow v = -1.$

③: $u^2+3v = u^2 - 3 = 1 \Rightarrow u^2 = 4 \Rightarrow u = \pm 2$

$(-2,-1)$: $\begin{cases} (-2)^2 - (-1)^2 = 3 & ✓ \\ -2-1 = -3 & X \end{cases}$ $\quad (-2,-1)$ doesn't work.

$(2,-1)$: $\begin{cases} 2^2 - (-1)^2 = 3 & ✓ \\ 2+(-1) = 1 & ✓ \\ 2^2 + 3(-1) = 1 & ✓ \end{cases}$ $\quad (2,-1)$ is the point we are looking for.

$\vec{r}_u = \langle 2u, 1, 2u \rangle, \quad \vec{r}_u(2,-1) = \langle 4, 1, 4 \rangle$

$\vec{r}_v = \langle -2v, 1, 3 \rangle, \quad \vec{r}_v(2,-1) = \langle 2, 1, 3 \rangle$

$$\vec{r}_u(2,-1) \times \vec{r}_v(2,-1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 1 & 4 \\ 2 & 1 & 3 \end{vmatrix} = \langle -1, -4, 2 \rangle$$

So, the tangent plane at $(3,1,1)$ is

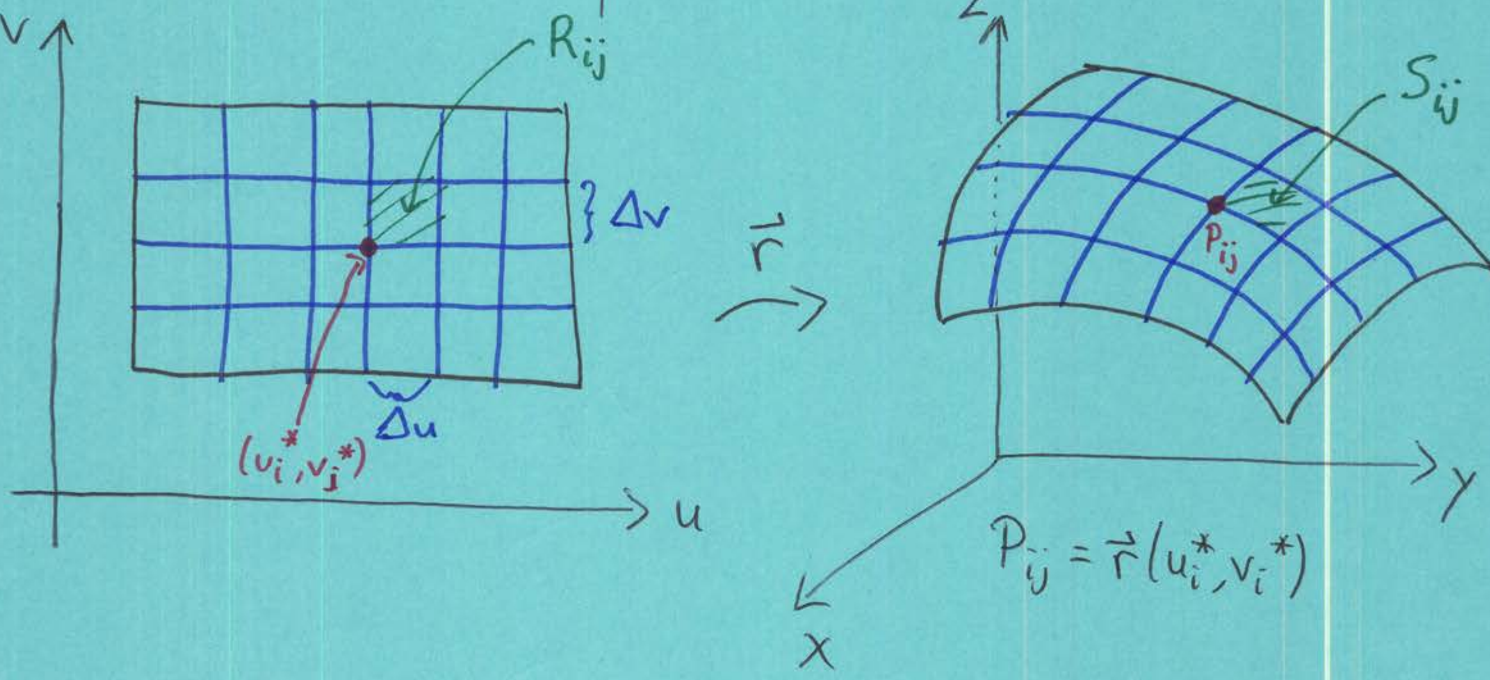
$$\langle -1,-4,2 \rangle \cdot \langle x-3, y-1, z-1 \rangle = \boxed{-(x-3) - 4(y-1) + 2(z-1) = 0} \qquad \diamond$$

Another question we can ask about surfaces is "what is the area of a surface?"

Let S be a surface parametrized by

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle, \quad (u,v) \in D.$$

We have, in a simple case:



Let ΔS_{ij} = area of parallelogram over S_{ij}, tangent at P_{ij}.

Then: $\Delta S_{ij} = |\vec{r}_u(u_i^*, v_j^*) \times \vec{r}_v(u_i^*, v_j^*)| \, \Delta u \Delta v$

So, the surface area of S is:

$$\boxed{A(S) = \boxed{\lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \Delta S_{ij}} = \iint_S dS = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA}$$

Ex: Find the surface area of the cylinder described by $x^2+y^2=4$, $0\le z\le 3$.

Sol: First, we parametrize: use cylindrical coordinates since $r=2$ here.

$$\vec{r}(\theta,z)=\langle 2\cos\theta, 2\sin\theta, z\rangle,\ 0\le\theta\le 2\pi,\ 0\le z\le 3.$$

$$\vec{r}_\theta=\langle -2\sin\theta, 2\cos\theta, 0\rangle,\ \vec{r}_z=\langle 0,0,1\rangle.$$

$$\vec{r}_\theta\times\vec{r}_z=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -2\sin\theta & 2\cos\theta & 0\\ 0 & 0 & 1\end{vmatrix}=\langle 2\cos\theta, 2\sin\theta, 0\rangle.$$

$$|\vec{r}_\theta\times\vec{r}_z|=2.$$

$$A(S)=\iint_S dS=\iint_D|\vec{r}_\theta\times\vec{r}_z|\,dA=\int_0^3\int_0^{2\pi}2\,d\theta\,dz$$

$$=\int_0^3 4\pi\,dz=12\pi.$$

$\diamond$

In the special case of a surface which is the graph of a function: $z=f(x,y)$, $(x,y)\in D$, the surface area formula takes a special form:

$$\vec{r}(x,y)=\langle x, y, f(x,y)\rangle$$

$$\vec{r}_x=\langle 1,0,f_x(x,y)\rangle,\quad \vec{r}_y=\langle 0,1,f_y(x,y)\rangle$$

$$\vec{r}_x\times\vec{r}_y=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 0 & f_x\\ 0 & 1 & f_y\end{vmatrix}=\langle -f_x, -f_y, 1\rangle$$

$$\Rightarrow |\vec{r}_x\times\vec{r}_y|=\sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+1}$$

$$\Rightarrow A(S)=\iint_D \sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+1}\,dA$$

Which looks familiar to the arc length formula from Calc II: $L=\int_a^b \sqrt{[f'(x)]^2+1}\,dx$ for a curve which is the graph of a function $y=f(x)$ on $[a,b]$.

(Personally, I find it a waste of time to memorize this formula when the previous method works just as well.)